Due: **11:59pm EST, Sunday March 12, 2023**

MAT185 Linear Algebra
Assignment 3

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
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2) ______

1. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Suppose A_1 and A_2 are subspaces of V.

Read and then write a critique the following "proof" that $T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$. The proof consists of five lines, not including assumptions. Your critique should identify, at a minimum, which lines are not correct; where exactly the proof breaks down; and what exactly are the incorrect statements or deductions.

"Proof":

Suppose that $\mathbf{x} \in T(A_1 \cap A_2)$.

Line 1: Then there exists a vector $\mathbf{y} \in A_1 \cap A_2$ such that $T\mathbf{y} = \mathbf{x}$.

Line 2: Since $\mathbf{y} \in A_1$ and $\mathbf{y} \in A_2$, we have $T\mathbf{y} \in T(A_1)$ and $T\mathbf{y} \in T(A_2)$, so that $\mathbf{x} \in T(A_1) \cap T(A_2)$.In other words, we have shown that $T(A_1 \cap A_2) \subseteq T(A_1) \cap T(A_2)$.

Now suppose that $\mathbf{x} \in T(A_1) \cap T(A_2)$.

Line 3: Then there exists a vector $\mathbf{y} \in A_1$ such that $T\mathbf{y} = \mathbf{x}$ and there exists a $\mathbf{y} \in A_2$ such that $T\mathbf{y} = \mathbf{x}$.

Line 4: But, $\mathbf{y} \in A_1 \cap A_2$ so that $\mathbf{x} \in T(A_1 \cap A_2)$. In other words, we have shown that $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$.

Line 5: Since we have shown both $T(A_1 \cap A_2) \subseteq T(A_1) \cap T(A_2)$, and $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$ we have $T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$.

After examining the proof, lines 3 and 4 make incorrect assumptions and deductions, and therefore the proof breaks down at line 3 and 4.

In line 3, the proof assumes that the transformation $T : V \to W$ is injective. (i.e. the transformation is one to one). It basically implies that only one vector $y \in A_1$ and $y \in A_2$ exists such that $Ty = x$. However, this is a false/incorrect assumption, as there can be more than one vector in A_1 where $Ty = x$ and more than one vector in A_2 where $Ty = x$. Hence this assumption is incorrect.

Line 4 makes an incorrect deduction from the wrong assumption made earlier in line 3. For instance, due to the fact that more than one vector in A_1 can be mapped to $x \in T(A_1) \cap T(A_2)$, it means that there can be a vector $y \in A_1$ but not in $A_1 \cap A_2$ that can be mapped by $Ty = x$. Similarly there can also be a vector $y \in A_2$ that is not in $A_1 \cap A_2$ for which $Ty = x$. As a result, every vector $y \in A_1$ that gets mapped by $Ty = x$ and every vector $y \in A_2$ that gets mapped by $Ty = x$ may not necessarily be in $y \in A_1 \cap A_2$. As such y is not always in $A_1 \cap A_2$ and therefore the beginning of line 4 is wrong. Correspondingly the statement $x \in T(A_1 \cap A_2)$ cannot be deduced. Therefore Line 4 doesn't hold, and we cannot conclude that $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$

2. Let $c \in \mathbb{R}$, and let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = cp(x) - xp'(x)$.

Determine all values of c such that T is bijective?

If T is an isomorphism (i.e. bijective), then it is both injective and surjective by definition. To determine the values of c such that T is bijective, consider a polynomial p(x) of degree n, where:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Then $$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1}$$

$$xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + \ldots + n a_n x^n$$

where $a_1, a_2, \ldots a_n$ are all real coefficients.

We know T will be surjective for any value of c ($c \in \mathbb{R}$):

$\operatorname{Im} T = W$

$$\begin{aligned} T(p(x)) &= cp(x) - xp'(x) \\ &= c(a_0 + a_1 x + \ldots + a_n x^n) - (a_1 x + 2a_2 x + \ldots + n a_n x^n) \\ &= ca_0 + (c-1)a_1 x + (c-2)a_2 x^2 + \ldots + (c-n)a_n x^n \\ &= \sum_{n=0}^{\infty} (c-n) a_n x^n \end{aligned}$$

To determine the conditions on c, we need to determine the values of c for which T is not injective ($\exists\, p(x) \neq 0$ such that $T(p(x)) = 0$).

For T to be non-injective, we can consider non-zero monomials (i.e polynomials consisting of single terms) of degrees n such that $T(P(x)) = 0$, for <u>any non-zero p(x)</u>.

- 0-degree monomial $p(x) = a_0$ $\quad T(p(x)) = ca_0 = 0 \longrightarrow c = 0$
- 1st-degree monomial $p(x) = a_1 x$ $\quad T(p(x)) = ca_1 x - a_1 x = (c-1)a_1 x = 0 \longrightarrow c = 1$
- 2nd-degree monomial $p(x) = a_2 x^2$ $\quad T(p(x)) = ca_2 x^2 - 2a_2 x^2 = (c-2)a_2 x^2 = 0 \rightarrow c = 2$
- 3rd-degree monomial $p(x) = a_3 x^3$ $\quad T(p(x)) = ca_3 x^3 - 3a_3 x^3 = (c-3)a_3 x^3 = 0 \rightarrow c = 3$
- $\vdots \qquad \vdots \qquad \vdots \qquad \vdots$
- n^{th}-degree monomial $p(x) = a_n x^n$ $\quad T(p(x)) = ca_n x^n - na_n x^n = (c-n)a_n x^n = 0 \rightarrow c = n$

where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ are non-zero coefficients (i.e. $p(x) \neq 0$)

By considering non-zero monomials such that $T(P(x)) = 0$, we know that for $c = 0, 1, 2, 3, \ldots, n$, there exists non-zero p(x) for which $T(p(x)) = 0$. This simply means that for $c = 0, 1, 2, 3, \ldots n$, T is not injective (and therefore not bijective).

Hence, we can conclude, that in order for T to be bijective, $c \in \mathbb{R}$, where $c \neq 0, 1, 2, 3, \ldots, n$.

3. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v_3}$ be a basis for V.

(a) Prove that if T is bijective, then $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis for W.

If T is bijective, it is injective and surjective
According to the definition of bases, $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\} \in V$ is a basis for V iff it is linearly independent and spans V.

Suppose $\underset{\sim}{y} \in W$. $\Rightarrow \exists$ some $\underset{\sim}{x} \in V$ such that $T(\underset{\sim}{x}) = \underset{\sim}{y}$
Since $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ spans V, $\underset{\sim}{x}$ can be described as a linear combination of the span,

$$\underset{\sim}{x} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 \underset{\sim}{v}_3 \text{, where } c_1, c_2, c_3 \in \mathbb{R}$$

Because T is surjective:

$$\underset{\sim}{y} = T(\underset{\sim}{x}) = c_1 T\underset{\sim}{v}_1 + c_2 T\underset{\sim}{v}_2 + c_3 T\underset{\sim}{v}_3 \text{, where } c_1, c_2, c_3 \in \mathbb{R}.$$

Thus, since $\underset{\sim}{x} \in \text{span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$, $\underset{\sim}{y} = T(\underset{\sim}{x}) \in \text{span}\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$

According to the definition of bases, $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ is linearly independent
Thus $c_1 T\underset{\sim}{v}_1 + c_2 T\underset{\sim}{v}_2 + c_3 T\underset{\sim}{v}_3 = 0$ iff $c_1 = c_2 = c_3 = 0$.
According to linearity,

$$T(c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 \underset{\sim}{v}_3) = 0$$

Since T is injective:

$$c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 \underset{\sim}{v}_3 = 0. \text{ iff } c_1 = c_2 = c_3 = 0$$

Thus, because $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ is linearly independent, and because T is injective, $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ is linearly independent.

$\therefore$ Because T is bijective, $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ spans W and is linearly independent. Thus $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ is a basis for W

3. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis for V.

(b) Prove that if $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis for W, then T is bijective.

If T is bijective, it is injective and surjective.
According to the definition of bases $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ is a basis for W iff it spans W and is linearly independent.

Suppose $y \in W$. $\Rightarrow \exists$ some $\underset{\sim}{x} \in V$ such that $T(\underset{\sim}{x}) = y$
Since $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ spans W, $y \in W$ can be written as a linear combination of the basis:

$$y = c_1 T\underset{\sim}{v}_1 + c_2 T\underset{\sim}{v}_2 + c_3 Tv_3, \text{ where } c_1, c_2, c_3 \in \mathbb{R}$$

$$y = T(c_1 \underset{\sim}{v}_1, c_2 \underset{\sim}{v}_3, c_3 \underset{\sim}{v}_3) = T(\underset{\sim}{x})$$

Because $\underset{\sim}{x} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + c_3 \underset{\sim}{v}_3$, where $c_1, c_2, c_3 \in \mathbb{R}$,
T is surjective.

If T is injective, the dimension formula states that,

$$\dim \operatorname{Ker}(T) + \dim \operatorname{im}(T) = \dim V \quad \text{Thus, } \operatorname{Ker}(T) = \{\underset{\sim}{0}\}$$

Suppose $T(\underset{\sim}{x}) = \underset{\sim}{0}$

$$\underset{\sim}{x} = c_1 \underset{\sim}{v}_1, c_2 \underset{\sim}{v}_2, c_3 \underset{\sim}{v}_3, \text{ where } c_1, c_2, c_3 \in \mathbb{R}$$

In order for T to be injective:

$$T(\underset{\sim}{x}) = c_1 T\underset{\sim}{v}_1 + c_2 T\underset{\sim}{v}_2 + c_3 T\underset{\sim}{v}_3 = \underset{\sim}{0}, \text{ where } c_1, c_2, c_3 \in \mathbb{R}.$$

Since $\{T\underset{\sim}{v}_1, T\underset{\sim}{v}_2, T\underset{\sim}{v}_3\}$ is the basis for W, it is linearly independent.
So, therefore $T(\underset{\sim}{x}) = \underset{\sim}{0}$ iff $c_1, c_2, c_3 = 0$. So, $\underset{\sim}{x} = \underset{\sim}{0}$.
As a result, $\operatorname{Ker}(T) = \{\underset{\sim}{0}\}$ and $\dim \operatorname{Ker}(T) = 0$, so T is injective.

$\therefore$ T is surjective and injective. Thus T is bijective